Quantum field theory in curved spacetime

Assignment 9/Exam 2 – June 30

Please hand in this assignment before the tutorial at 14h15AM on June 30. In total, you need to obtain 40% of the combined points from the first and this exam.

Exercise 19: Euler-Heisenberg Lagrangian on a background of constant curvature

Motivation: Two weeks ago, we computed the full Euler-Heisenberg Lagrangian on a flat background. This time, we include background curvature and use heat kernels as we learned last week, and only consider the first corrections in curvature and electromagnetic field. We'll find that nonminimal coupling of electromagnetism to gravity is unavoidable!

The Euler-Heisenberg effective action on a curved background satisfies

$$e^{i\Gamma[A,g]} = e^{iS_{EH}[A,g]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QED}}[A,g]}, \tag{19.1}$$

with the QED action on a curved background and the Einstein-Hilbert action

$$S_{\mathrm{QED}} = \int \mathrm{d}^4x\sqrt{-g}\left[-\mathcal{F} + \bar{\psi}\left(i\not{D} - m\right)\psi\right], \tag{19.2}$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R - 2\Lambda\right), \tag{19.3}$$

respectively. Here, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu}$ is the field-strength tensor of the gauge field A_μ, and R and Λ denote the Ricci scalar and the cosmological constant. Besides, $\not{D}$ denotes the covariant Dirac operator involving the covariant derivative including the spin connection $\omega_{\mu ab}$

$$D_\mu\psi = (\nabla_\mu + ieA_\mu)\psi = \left(\partial_\mu + ieA_\mu - i\omega_{\mu ab}\Sigma^{ab}\right)\psi, \tag{19.4}$$

where $\Sigma^{ab} = i[\gamma^a, \gamma^b]/8$ (note the different notation to exercise 16 where we used $\sigma^{ab} = i[\gamma^a, \gamma^b]/2$), local Lorentz indices are Latin, spacetime indices Greek letters, and e and m are the charge and the mass of the (Grassmann-valued) fermion ψ, respectively. For simplicity, we assume the background electromagnetic field strength to be constant as two weeks ago.

(a) Discuss why a constant field strength necessarily means $\partial_c F_{ab} = 0$, *not* $\partial_\rho F_{\mu\nu} = 0$.

We want to compute the one-loop effective Lagrangian by integrating out the fermion. From exercise 16, we know that we can express the one-loop contribution to the effective action of the background fields as

$$\Gamma^{(1)}[A,g] = -\frac{i}{2}\log\det(\not{D}^2 + m^2), \tag{19.5}$$

$$= \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\,\mathrm{tr}\left(e^{-sH}\right) + \mathrm{const}, \tag{19.6}$$

for the Hamiltonian $H = \not{D}^2$.

(**b**) Two weeks ago, we just ignored the (infinite) constant contributions to the effective action. Discuss whether we can still do that. What do they contribute to? Hereafter, we assume to have dealt with those contributions successfully.

(**c**) Demonstrate that the Hamiltonian can be expressed as

$$H = D^2 + 2eF_{ab}\Sigma^{ab} - \frac{R}{4}. \tag{19.7}$$

Hint: You can use without proof that the Lorentz generators satisfy the Lorentz algebra

$$[\Sigma^{ab}, \Sigma^{cd}] = i\left(\eta^{c[a}\Sigma^{b]d} - \eta^{d[a}\Sigma^{b]c}\right). \tag{19.8}$$

Besides the Riemann tensor with Lorentz indices $R_{abcd} = e^{\mu}_{a}e^{\nu}_{b}R_{\mu\nu cd}$ has the same symmetries as the usual Riemann tensor, and satisfies the first Bianchi identity.

(**d**) Discuss why we can separate the traces as

$$\Gamma^{(1)}[A, g] = \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\operatorname{tr}\left(e^{-sH_{\text{kin}}}\right)\operatorname{tr}\left(e^{-sH_{\text{spin}}}\right), \tag{19.9}$$

with $H_{\text{kin}} = D^2 - \frac{R}{4}$, and $H_{\text{spin}} = 2eF_{ab}\Sigma^{ab}$ We have already computed the spin trace in flat space. Explain why this result remains valid and we can copy it to get

$$\operatorname{tr}\left(e^{-sH_{\text{spin}}}\right) = 4\cos(esa)\cosh(esb), \tag{19.10}$$

where

$$a^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}, \qquad b^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}. \tag{19.11}$$

As in exercise 16, here we defined $\mathcal{G} \equiv F_{ab}\tilde{F}^{ab}/4$, and the dual field strength $\tilde{F}^{ab} = \epsilon^{abcd}F_{cd}/2$. Let me remind you that $a = 0$ if $\vec{E} = 0$ and $b = 0$ if $\vec{B} = 0$, so we can understand a to largely measure electric contributions to the field strength, while b largely measures the magnetic ones.

Time to compute $\operatorname{tr} e^{-sH_{\text{kin}}}$. Here, we get to the promised heat kernels. The cool thing about the heat-kernel method is that it does not only work with the Levi-Civita connection; any gauge connection will do. Like last week, we define

$$\frac{\mathrm{d}}{\mathrm{d}s}K(\mathbf{x}, \mathbf{x}', s) = -\left(\mathcal{D}^2 + \mathcal{E}\right)K(\mathbf{x}, \mathbf{x}', s), \tag{19.12}$$

where for the moment $\mathcal{D}_\mu = \partial_\mu + i\mathcal{A}_\mu$ is some differential operator with a gauge connection $\mathcal{A}_\mu$ possibly having suppressed internal indices, while $\mathcal{E}$ is a so-called endomorphism – basically a possibly matrix-valued function of the position like a potential, the Ricci scalar or H_{spin} (above we took that out of the trace, but we can't do so for inhomogeneous electromagnetic fields). Under a gauge transformation $U(x)$, the covariant derivative, the endomorphism and the metric generally transform

$$\mathcal{D}_\mu \to \mathcal{D}^U_\mu = U\mathcal{D}_\mu U^{-1}, \qquad \mathcal{E} \to \mathcal{E}_U = U\mathcal{E}U^{-1}, \qquad g^{\mu\nu} \to g^{\mu\nu}_U = Ug^{\mu\nu}U^{-1}. \tag{19.13}$$

If the gauge transformation concerns internal degrees of freedom, *i. e.* for all gauge transformations but diffeomorphisms, $[g^{\mu\nu}, U] = 0$ such that $g^{\mu\nu}_U = g^{\mu\nu}$.

(**e**) Show that the operator $K_U = UKU^{-1}$ satisfies the heat equation with respect to the transformed operator $(\mathcal{D}^U)^2 + \mathcal{E}^U$. Discuss why this implies that $\int \sqrt{-g}\mathrm{d}^4xK(x,x,s)$ is a gauge invariant quantity.

(**f**) Let's go back to the specific case in this exercise. We make the ansatz

$$K(x,x,s) = \frac{1}{(4\pi s)^2}\sum_{n=0}^{\infty} a_n(x)s^n. \tag{19.14}$$

To order $n = 1$, we already computed the coefficients $a_0 = 1$, $a_1 = R/6$ last week. Explain why there can't be any new contributions from electromagnetism to that order.

(**g**) Write down the contributions containing the electromagnetic field strength you expect to appear in a_2 and a_3. Explain what those contributions mean.

Hint: Follow the concept that everything that is not prohibited is compulsory.

(**h**) If electromagnetism is nonminimally coupled to gravity, there are a lot of new effects – for example, photons need not move on null geodesics any more, they may perceive an effective spacetime, which is not just governed by $g_{\mu\nu}$. Explain why we do not measure these effects even though they are presumably present.

Exercise 20: Weyl anomaly for a scalar field

Motivation: In the lecture, we already encountered the conformal anomaly for fermions. Now, we have a look at scalar fields.

Consider a real scalar field ϕ whose dynamics is governed by the action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{R\phi^2}{6}\right). \tag{20.1}$$

We know from assignment 2 that the action is invariant under the Weyl transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}, \qquad \phi \to \tilde{\phi} = \Omega(x)^{-1}\phi. \tag{20.2}$$

In this exercise, we want to find out whether the conformal symmetry survives for a quantum scalar field using Fujikawa's method. The quantum scalar field is governed by the path integral

$$Z = e^{i\Gamma} = \int \mathcal{D}\phi e^{iS}. \tag{20.3}$$

As S is invariant, we have to look at the measure $\mathcal{D}\phi$. Let's put our system into a box of length L. Then, the Weyl-invariant Klein-Gordon operator has a discrete spectrum, and we define its eigenfunctions ϕ_n as

$$\left(\Box + \frac{R}{6}\right)\phi_n = \lambda_n\phi_n, \tag{20.4}$$

where $\lambda_n \in \mathbb{R}$. These eigenfunctions are orthonormal and complete, *i. e.*

$$\int \mathrm{d}^4x\sqrt{-g}\phi_n^*(x)\phi_{n'}(x) = L^2\delta_{nn'}, \tag{20.5}$$

$$\sum_n \phi_n^*(x)\phi_n(x') = L^2\frac{\delta(x-x')}{\sqrt{g}}. \tag{20.6}$$

Besides, they can be chosen such that they are real, *i. e.* $\phi_n^* = \phi_n$, which we do hereafter. Thus, we can express any field ϕ included in the configuration space as

$$\phi = \sum_n c_n \phi_n \tag{20.7}$$

for some set of coefficients c_n.

(**a**) Define the path integral measure such that the effective action takes the form

$$\Gamma = \frac{i}{2} \log \det L^2 \left(\Box + \frac{R}{6} \right). \tag{20.8}$$

This is what we want the effective action formally to look like.

(**b**) Demonstrate that after a Weyl transformation – we choose the ϕ_n such that Eq. (20.2) implies $\phi_n \to \tilde{\phi}_n = \Omega^{-1}\phi_n$ – the eigenfunctions are neither orthogonal nor normalized anymore. Instead, show that the functions $\phi_n^\Omega = \Omega^{-1}\tilde{\phi}_n$ are orthonormal in the Weyl-transformed spacetime.

(**c**) Consider the expansion

$$\tilde{\phi} \equiv \sum_n c_n^\Omega \phi_n^\Omega. \tag{20.9}$$

Construct the corresponding measure in the Weyl-transformed spacetime and show that the two measures are related by a Jacobian

$$\mathcal{D}\phi = \mathcal{D}\tilde{\phi} \det \left(\frac{\partial c_n}{\partial c_{n'}^\Omega} \right), \tag{20.10}$$

where the determinant is taken over the index space n, n'.

(**d**) Consider an infinitesimal transformation $\Omega(x) = 1 + \omega(x)$. Compute the Jacobian. You should obtain

$$\frac{\partial c_n}{\partial c_{n'}^\Omega} = \delta_{nn'} - L^2 \int \mathrm{d}^4x \sqrt{-g}\, \omega \phi_n \phi_{n'}. \tag{20.11}$$

(**e**) Define

$$Z^\Omega = e^{i\Gamma^\Omega} \equiv \int \mathcal{D}\phi^\Omega e^{iS}, \tag{20.12}$$

to show that

$$\delta\Gamma \equiv \Gamma^\Omega - \Gamma = i \log \det \left(\frac{\partial c_n}{\partial c_{n'}^\Omega} \right) = -L^2 \sum_n \int \mathrm{d}^4x \sqrt{-g}\, \omega \phi_n^2. \tag{20.13}$$

(**f**) Recall that the effective action is the quantum equivalent of an ordinary action. Analogously to the derivation of the classical stress-energy tensor from the classical action, one can (you don't need to) show that

$$\delta\Gamma = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} \langle T_{\mu\nu} \rangle \delta g^{\mu\nu}. \tag{20.14}$$

Demonstrate that

$$\langle T^\mu_\mu(x) \rangle = L^2 \sum_n \phi_n^2(x). \tag{20.15}$$

(**g**) Discuss what happens if we naively apply the completeness relation, Eq. (20.6), to Eq. (20.15). To resolve this issue will be the subject of the next assignment.